An intertwined system of recurrences related to the golden mean

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In his well–known book "Gödel, Escher, Bach: An Eternal Golden Braid", D. R. Hofstadter introduces a "married couple" of integer functions $M(n)$, $F(n)$ defined by $M(0) = 0$, $F(0) = 1$ and

$$\begin{aligned} M(n) &= n - F(M(n-1)) \\ F(n) &= n - M(F(n-1)) \quad \text{for } n > 0. \end{aligned}$$

We first show that $F(n) = \lfloor (n+1)\mu \rfloor + \varepsilon_1$ and $M(n) = \lfloor (n+1)\mu \rfloor - \varepsilon_2$ with $\varepsilon_1, \varepsilon_2 \in \{0,1\}$ and $\mu = \phi^{-1}$, where $\phi = (\sqrt{5}+1)/2$ is the golden mean. In generalizing this result, we consider the intertwined system of $N \geq 3$ recurrences

$$\begin{aligned} a_1(n) &= n - a_N(a_1(n-1)) \\ a_2(n) &= n - a_1(a_2(n-1)) \\ &\vdots \\ a_k(n) &= n - a_{k-1}(a_k(n-1)) \\ &\vdots \\ a_N(n) &= n - a_{N-1}(a_N(n-1)) \end{aligned}$$

with $a_k(0) = 0$ for $1 \leq k \leq N$, $k \neq K$ and $a_K(0) = 1$ for some $2 \leq K \leq N$. Again, we obtain some explicit formulæ for $a_j(n)$ which involve both the golden ratio and Fibonacci numbers. In contrast to the original case, irregular oscillations occur.

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