On norm form equations with solutions forming arithmetic progressions

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Let $\alpha_1 = 1, \alpha_2, \ldots, \alpha_m$ be linearly independent algebraic numbers over $\mathbb{Q}$ and put $K := \mathbb{Q}(\alpha_1, \ldots, \alpha_m)$. Let $n := [K : \mathbb{Q}]$. For any $\alpha \in K$, denote by $\alpha^{(i)}$ the conjugates of α. Put $l^{(i)}(\mathbf{X}) = X_1 + \alpha_2^{(i)} X_2 + \cdots + \alpha_n^{(i)} X_n$ for $i = 1, \ldots, n$. There exists a non-zero $a_0 \in \mathbb{Z}$ such that the form $F(\mathbf{X}) := a_0 N_{K/\mathbb{Q}}(\alpha_1 X_1 + \cdots + \alpha_m X_m) = a_0 \prod_{i=1}^n l^{(i)}(\mathbf{X})$ has integer coefficients. Such a form is called a **norm form**.
The equation

$$a_0 N_{K/\mathbb{Q}}(\alpha_1 x_1 + \cdots + \alpha_m x_m) = b \tag{1}$$

in $x_1, \ldots, x_m \in \mathbb{Z}$ is called a **norm form equation**.
If the $\mathbb{Q}$ vector space spanned by $\alpha_1, \ldots, \alpha_m$ has a subspace, which is proportional to a full $\mathbb{Z}$-module of an algebraic number field, different from $\mathbb{Q}$ and the imaginary quadratic field, then $\alpha_1\mathbb{Z} + \cdots + \alpha_m\mathbb{Z}$ is called degenerate. Then it is easy to see, that (1) can have infinitely many solutions.
Buchmann and Pethő found twenty years ago, as a byproduct of a search for independent units that in the field $K := \mathbb{Q}(\alpha)$ with $\alpha^7 = 3$, the integer $10 + 9\alpha + 8\alpha^2 + 7\alpha^3 + 6\alpha^4 + 5\alpha^5 + 4\alpha^6$ is a unit. This means that the diophantine equation

$$N_{K/\mathbb{Q}}(x_0 + x_1\alpha + \cdots + x_6\alpha^6) = 1 \tag{2}$$

has a solution $(x_0, \ldots, x_6) \in \mathbb{Z}^7$ such that the coordinates form an arithmetic progression.
Our goals are: Generalize (2) in three directions, and investigate those solutions which form an arithmetic progression:

- we consider arbitrary number fields,
- the integer on the right hand side of equation (2) is not restricted to 1,
- it is allowed that the solutions form only nearly an arithmetic progression.

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